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INTEGRAL MEANS OF UNIVALENT FUNCTIONS WITH RESTRICTED HAYMAN INDEX

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ABSTRACT

Kamotskii proved that if $f \in S$ has Hayman index α then, for $p = 1, 2, 3$,

$$(1) \qquad \int_{-\pi}^{\pi} |f(re^{it})|^p dt \geqq \int_{-\pi}^{\pi} |\alpha k(re^{it})|^p dt, \qquad 0 < r < 1.$$

In this paper we extend Kamotskii's result. We prove that if f has Hayman index α and its logarithmic coefficients γ_n satisfy $|\gamma_n| \leqq 1/n$ for all n, then, for $0 < r < 1$,

$$(2) \qquad \int_{-\pi}^{\pi} \phi(\log|f(re^{it})|)dt \geqq \int_{-\pi}^{\pi} \phi(\log|\alpha k(re^{it})|)dt$$

for all convex increasing functions $\phi(x)$ defined on $\mathbf{R}$ and give examples which show that this is not true in general. Also, we prove that (1) remains true for $2 < p < 3$ and general f and for $p = 4$ if f has real Taylor coefficients.

1. Introduction

Let D denote the unit disc $\{z : |z| < 1\}$. For $p > 0$ and g analytic in D, define

$$M_p(r, g) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{it})|^p dt\right)^{1/p}, \qquad 0 < r < 1,$$

$$M_\infty(r, g) = \operatorname*{Max}_{|z|=r} |g(z)|, \qquad 0 < r < 1.$$

Let S be the class of functions f analytic and univalent in D with

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$f(0)=0$, $f'(0)=1$. Hayman [7, 8] (see also [5, pp. 157–158]) proved that if $f\in S$ then the limit

$$\alpha=\lim_{r\to 1} r^{-1}(1-r)^2 M_\infty(r,f) \tag{1}$$

exists and $0\leqq\alpha\leqq 1$; furthermore, $\alpha=1$ if and only if f is a rotation of the Koebe function $k(z)=z/(1-z)^2$. If $\alpha>0$ then f has a unique direction of maximal growth, i.e. there exists a unique $\theta_0\in[0,2\pi)$ such that $(1-r)^2|f(re^{i\theta_0})|\to\alpha$, as $r\to 1$. In fact, we have

$$|f(re^{i\theta_0})|\geqq\alpha k(r),\qquad 0<r<1, \tag{2}$$

$$|f(re^{i\theta_0})|\sim\alpha k(r),\qquad \text{as } r\to 1. \tag{3}$$

The number α is called the Hayman index of f and we will denote by $S(\alpha)$ the class of those $f\in S$ whose Hayman index is α.

Hayman [7, 8] proved that if $f(z)=\Sigma\, a_n z^n\in S(\alpha)$ then $|a_n|/n\to\alpha$. Milin obtained in [12] (see also [5, pp. 162–166]) a simplified proof of this result which is known as Hayman's regularity theorem.

In [1] Baernstein proved that the Koebe function is extremal for a large class of problems about integral means in the class S. He proved that if $f\in S$ then

$$\int_{-\pi}^{\pi}\phi(\pm\log|f(re^{it})|)dt\leqq\int_{-\pi}^{\pi}\phi(\pm\log|k(re^{it})|)dt,\qquad 0<r<1, \tag{4}$$

for all convex increasing functions $\phi(x)$ defined on $\mathbf{R}$. In particular, for $0<p<\infty$, the Koebe function has the largest L^p-means among all the functions in the class S.

Kamotskii studied in [11] the integral means of functions in the classes $S(\alpha)$. He proved that if $f\in S(\alpha)$ then, for $p=1,2,3$,

$$M_p(r,f)\geqq M_p(r,\alpha k),\qquad 0<r<1. \tag{5}$$

The purpose of this paper is studying whether or not Kamotskii's result can be generalized to cover other integral means. First we study the possibility of this generalization in the direction of (4): Let $f\in S(\alpha)$, is it true that, for $0<r<1$,

$$\int_{-\pi}^{\pi}\phi(\log|f(re^{it})|)dt\geqq\int_{-\pi}^{\pi}\phi(\log|\alpha k(re^{it})|)dt$$

for all convex increasing functions $\phi(x)$ defined on $\mathbf{R}$? We will show that the answer is affirmative if we impose some restrictions on f but not in general. Then, in section 3, we will prove (5) for some other values of p.

2. Baernstein type inequalities

Associated with each function f in S are its logarithmic coefficients γ_n defined by

$$\log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n, \qquad z\in D. \tag{6}$$

The logarithmic coefficients of the Koebe function are $\gamma_n = 1/n$. We prove

THEOREM 1. *Let $0<\alpha\leqq 1$ and $f\in S(\alpha)$. If the logarithmic coefficients γ_n of f satisfy $|\gamma_n|\leqq 1/n$ for all n, then*

$$\int_{-\pi}^{\pi}\phi(\log|f(re^{it})|)dt \geqq \int_{-\pi}^{\pi}\phi(\log|\alpha k(re^{it})|)dt, \qquad 0<r<1, \tag{7}$$

for all convex increasing functions $\phi(x)$ defined on $\mathbf{R}$.

PROOF. We may assume without loss of generality that $\theta_0 = 0$ is the direction of maximal growth of f. Thus,

$$|f(r)| \geqq \alpha k(r), \qquad 0<r<1. \tag{8}$$

Define

$$u(z) = \log\left|\frac{\alpha k(z)}{z}\right|, \quad v(z) = \log\left|\frac{f(z)}{z}\right| \qquad (z\in D). \tag{9}$$

According to [1, Prop. 3], the inequality (7) will hold provided we can show that

$$u^*(z) \leqq v^*(z), \qquad z\in D^+ = \{z\in D : \operatorname{Im} z>0\}, \tag{10}$$

where, for $0<r<1$ and $0<t<\pi$,

$$U^*(re^{it}) = \operatorname*{Sup}_{|E|=2t}\int_E U(re^{is})ds. \tag{11}$$

Notice that, since $|k(re^{it})|$ is a symmetric function of t on $[-\pi,\pi]$, decreasing on $[0,\pi]$,

$$u^*(re^{it}) = \int_{-t}^{t} u(re^{is})ds. \tag{12}$$

Define

$$v_*(re^{it}) = \int_{-t}^{t} v(re^{is})ds, \qquad 0 < r < 1, \quad 0 < t < \pi. \tag{13}$$

Then, since $v_* \leqq v^*$, (10) will follow from

$$u^*(z) \leqq v_*(z), \qquad z \in D^+. \tag{14}$$

To prove (14) we argue as follows. Define $V = u^* - v_*$. Then (12), (13) and the definitions of u and v show that

$$\begin{aligned} V(re^{it}) &= \int_{-t}^{t} \log\left|\frac{\alpha k(re^{is})}{f(re^{is})}\right| ds \\ &= \operatorname{Re} \int_{-t}^{t} \left(\log \alpha + 2 \sum_{k=1}^{\infty} \left(\frac{1}{k} - \gamma_k\right) r^k e^{iks}\right) ds \\ &= 2t \log \alpha + 4 \sum_{k=1}^{\infty} \left(\frac{1}{k} - \operatorname{Re} \gamma_k\right) r^k \frac{\sin kt}{k}. \end{aligned}$$

Thus, (14) is equivalent to

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \operatorname{Re} \gamma_k\right) \frac{r^k \sin kt}{k} \leqq \left(\frac{1}{2} \log \frac{1}{\alpha}\right) t, \qquad 0 < r < 1, \quad 0 < t < \pi. \tag{15}$$

But, since $|\gamma_k| \leqq 1/k$ and $|\sin x| \leqq |x|$, (8) implies

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \operatorname{Re} \gamma_k\right) r^k \frac{\sin kt}{k} &\leqq \sum_{k=1}^{\infty} \left(\frac{1}{k} - \operatorname{Re} \gamma_k\right) r^k \left|\frac{\sin kt}{k}\right| \\ &\leqq t \sum_{k=1}^{\infty} \left(\frac{1}{k} - \operatorname{Re} \gamma_k\right) r^k \\ &= \frac{t}{2} \log\left|\frac{k(r)}{f(r)}\right| \\ &\leqq \frac{1}{2}\left(\log \frac{1}{\alpha}\right) t, \qquad 0 < r < 1, \quad 0 < t < \pi. \end{aligned}$$

This proves (15) and finishes the proof of Theorem 1.

The inequality $|\gamma_n| \leqq 1/n$ $(n = 1, 2, 3, \ldots)$ is known to be true for starlike functions, however we should remark that, as Eenigenburg and Keogh proved [6, Th. 5], the rotations of the Koebe function are the only starlike functions with positive Hayman index and, hence, Theorem 1 says nothing for starlike functions.

The inequality $|\gamma_n| \leqq 1/n$ fails in general, even in order of magnitude. In fact, Pommerenke constructed in [14] (see also [15, Th. 5.4 and exer. 3, p. 138] and [5, Th. 8.4]) a bounded function $f \in S$ with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$.

Next, we shall give for each $\alpha \in (0, 1)$ a simple explicit function $f \in S(\alpha)$ for which the conclusion of Theorem 1 does not hold and, consequently, satisfying $|\gamma_n| > 1/n$ for some n.

For $\frac{1}{4} < d < 1$, let f_d be the unique function in S which maps D onto the whole w-plane slit along an arc on the circle $|w| = d$ placed symmetrically with respect to the negative real axis and the part of the negative real axis from $-d$ to $-\infty$. The functions f_d have been extensively treated in the literature. Netanyahu proved in [13] that f_d maximizes $|f''(0)|$ in the class S_d of functions $f \in S$ whose image contains the disc $|w| < d$ and Jenkins proved in [10] that the length of the set of values on $|w| = d$ not covered by f is maximized by f_d in the class S. We will show that the conclusion of Theorem 1 does not hold for the functions f_d, $\frac{1}{4} < d < 1$.

In order to give an explicit expression for f_d, it is convenient to define ρ by

$$d = \frac{(\rho + 1)^2}{4\rho^2}, \qquad \tfrac{1}{4} < d < 1. \tag{16}$$

Notice that as d increases from $\frac{1}{4}$ to 1, ρ decreases from ∞ to 1. Then, given $\rho > 1$, the function

$$F(z) = \left(\rho + \frac{1}{\rho} - 2\right)\left(\frac{1+z}{1-z}\right)^2 = \frac{(\rho - 1)^2}{\rho}\left(\frac{1+z}{1-z}\right)^2$$

maps D onto R_1, the whole plane slit along the negative real axis, and $F(0) = \rho + \rho^{-1} - 2$. The function

$$G(\xi) = \xi + \frac{1}{\xi} - 2 = \frac{(\xi - 1)^2}{\xi}$$

maps the domain R_2 bounded by $|\xi| = 1$ together with the portion of the real axis $-\infty \leqq \xi \leqq -1$ onto R_1 and $G(\rho) = \rho + \rho^{-1} - 2$. Finally, the function

$$H(\xi) = \frac{\rho\xi(\xi - \rho)}{\rho\xi - 1}$$

maps R_2 onto a domain R_3 bounded by an arc on $|w| = \rho$ placed symmetrically with respect to the negative real axis together with the portion $-\infty \leqq w \leqq -\rho$ of the latter and $H(\rho) = 0$. Then, the function $g = H \circ G^{-1} \circ F$ maps D conformally onto R_3 and $g(0) = 0$. Easy calculations show that $g'(0) = 4\rho^3/(\rho + 1)^2$ and $(1 - r)^2 g(r) \sim 4(\rho - 1)^2/\rho$, as $r \to 1$. Hence

$$f_d(z) = \frac{(\rho + 1)^2}{4\rho^3} g(z) \tag{17}$$

and the Hayman index α_d of f_d is

$$\alpha_d = \frac{(\rho - 1)^2(\rho + 1)^2}{\rho^4} = \left(1 - \frac{1}{\rho^2}\right)^2. \tag{18}$$

Notice that as d increases from $\frac{1}{4}$ to 1, α_d decreases from 1 to 0 and, hence, we see that for each $\alpha \in (0, 1)$ there exists exactly one d such that $\alpha_d = \alpha$. Now we can prove

THEOREM 2. *For* $\frac{1}{4} < d < 1$,

$$\int_{-\pi}^{\pi} \log^+ |f_d(e^{it})/d|\, dt < \int_{-\pi}^{\pi} \log^+ |\alpha_d k(e^{it})/d|\, dt. \tag{19}$$

Hence, the conclusion of Theorem 1 *does not hold for the functions* f_d, $\frac{1}{4} < d < 1$.

PROOF. Notice that $|f_d(e^{it})| \geqq d$ and, hence,

$$\int_{-\pi}^{\pi} \log^+ |f_d(e^{it})/d|\, dt = \int_{-\pi}^{\pi} \log |f_d(e^{it})/(de^{it})|\, dt = -2\pi \log d.$$

Now we turn to evaluate the integral on the right-hand side of (19):

$$\begin{aligned}\int_{-\pi}^{\pi} \log^+ \left|\frac{\alpha_d k(e^{it})}{d}\right| dt &= \int_{-\pi}^{\pi} \log^+ \left|\frac{\alpha_d}{4d \sin^2(t/2)}\right| dt \\ &= 4 \int_0^{\pi/2} \log^+ \frac{\alpha_d}{4d \sin^2 t}\, dt \\ &= 4 \int_0^{\arcsin(\alpha_d/4d)^{1/2}} \log \frac{\alpha_d}{4d \sin^2 t}\, dt \\ &= 4 \arcsin \left(\frac{\alpha_d}{4d}\right)^{1/2} \log \frac{\alpha_d}{4d} - 8 \int_0^{\arcsin(\alpha_d/4d)^{1/2}} \log \sin t\, dt.\end{aligned}$$

Now, (16) and (18) show that $(\alpha_d/4d) = [(\rho - 1)/\rho]^2$ and, hence, using (20) and (16), we see that (19) is equivalent to

$$\psi(\rho) > 0, \qquad 1 < \rho < \infty, \tag{21}$$

where

$$\psi(\rho) = 8 \arcsin \frac{\rho - 1}{\rho} \log \frac{\rho - 1}{\rho} - 8 \int_0^{\arcsin((\rho-1)/\rho)} \log \sin t \, dt + 4\pi \log \frac{\rho + 1}{2\rho} . \tag{22}$$

To prove (21), observe that

$$\psi(1) = 0, \qquad \lim_{\rho \to \infty} \psi(\rho) = 0 \tag{23}$$

and

$$\psi'(\rho) = \frac{8}{\rho(\rho - 1)} \left(\arcsin \frac{\rho - 1}{\rho} - \frac{\pi}{2} \frac{\rho - 1}{\rho + 1} \right). \tag{24}$$

Now, it is a simple calculus exercise to show that $\psi'(\rho) = 0$ at most once in $(1, \infty)$ which, together with (23) and the fact that $\psi'(\rho) > 0$ if ρ is sufficiently close to 1, implies (21), finishing the proof of Theorem 2.

3. L^p-inequalities

The basic step in the proof of Kamotskii's theorem is the following result proved by Kamotskii [11, p. 213] using the Grunsky inequalities:

Let $0 < \alpha \leqq 1$ and $f \in S(\alpha)$. Suppose that $\theta_0 = 0$ is the direction of maximal growth of f. Then, for all $z \in D$,

$$|f(z) - f(\bar{z})| \geqq \alpha |k(z) - k(\bar{z})| \tag{25}$$

and

$$|f_2(z) + f_2(\bar{z})|^2 \geqq \alpha |k_2(z) + k_2(\bar{z})|^2 \tag{26}$$

where $f_2(z) = f(z^2)^{1/2}$, $k_2(z) = k(z^2)^{1/2}$.

The inequality (5) follows from (25) for $p = 2, 3$ and from (26) for $p = 1$. Using (25), we shall obtain some extensions of Kamotskii's theorem.

Theorem 3. *Let $0 < \alpha \leqq 1$. If $f \in S(\alpha)$ has real Taylor coefficients then*

(27) $$M_4(r, f) \geqq M_4(r, \alpha k), \qquad 0 < r \leqq 1.$$

To prove Theorem 3 we will need the analogue of (2) for f'.

Proposition 1. *Let $0 < \alpha \leqq 1$. If $f \in S(\alpha)$ and θ_0 is its direction of maximal growth, then*

(28) $$|f'(re^{i\theta_0})| \geqq \alpha k'(r), \qquad 0 < r < 1.$$

Proposition 1 is obtained in [4, Cor. 1] as a consequence of Bazilevich's inequality on the closeness between the logarithmic coefficients of f and those of $z/(1 - e^{-i\theta_0} z)^2$,

$$\sum_{k=1}^{\infty} k \left| \gamma_k - \frac{1}{k} e^{-ik\theta_0} \right|^2 \leqq \frac{1}{2} \log \frac{1}{\alpha}$$

[2, 3] (see also [5, p. 160]) and it can also be obtained from (25). However, we present here an elementary proof.

Proof of Proposition 1. We may assume without loss of generality that $\theta_0 = 0$. For $0 < r < 1$, let

$$f_r(z) = \frac{f\left(\dfrac{z + r}{1 + rz}\right) - f(r)}{(1 - r^2) f'(r)}, \qquad z \in D.$$

Then $f_r \in S$ and, using (3), we see that, as $\rho \to 1$,

$$\begin{aligned}
(1 - \rho)^2 |f_r(\rho)| &\sim \frac{(1 - \rho)^2}{(1 - r^2)|f'(r)|} \left| f\left(\frac{\rho + r}{1 + \rho r}\right) \right| \\
&\sim \frac{(1 - \rho)^2}{(1 - r^2)|f'(r)|} \frac{\alpha}{\left(1 - \dfrac{\rho + r}{1 + \rho r}\right)^2} \\
&= \frac{\alpha(1 + \rho r)^2}{(1 - r^2)(1 - r)^2 |f'(r)|} \\
&\sim \frac{\alpha k'(r)}{|f'(r)|}.
\end{aligned}$$

Hence, the Hayman index $\alpha(r)$ of f_r is

$$\alpha(r) = \frac{\alpha k'(r)}{|f'(r)|}$$

which, since $\alpha(r) \leqq 1$, implies

$$|f'(r)| \geqq \alpha k'(r).$$

Proof of Theorem 3. Let $\theta_0 \in [0, 2\pi)$ be the direction of maximal growth of f. Then Hayman proved [7, p. 277] that

$$e^{-i\theta_0} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

where $f(z) = \Sigma\, a_n z^n$. Thus, since the coefficients a_n are real, we obtain that θ_0 is either 0 or π. If $\theta_0 = \pi$ then $h(z) = -f(-z)$ has real coefficients and its direction of maximal growth is 0. Consequently, we may assume without loss of generality that $\theta_0 = 0$.

The inequality (27) will follow from

$$\alpha^2 \int_{-\pi}^{\pi} |k(re^{it})|^4 dt \leqq \int_{-\pi}^{\pi} |f(re^{it})|^2 |k(re^{it})|^2 dt, \quad 0 < r < 1. \tag{29}$$

Indeed, by Hölder's inequality,

$$\int_{-\pi}^{\pi} |f(re^{it})|^2 |k(re^{it})|^2 dt \leqq M_4(r, f)^2 M_4(r, k)^2$$

and, hence, (29) implies

$$\alpha^2 M_4(r, k)^4 \leqq M_4(r, f)^2 M_4(r, k)^2, \qquad 0 < r < 1,$$

which is equivalent to (27).

Thus, it only remains to prove (29). It follows from (25) that, setting $z = re^{it}$,

$$\int_{-\pi}^{\pi} \frac{|f(z) - f(\bar{z})|^2}{|1 - z|^4} dt \geqq \alpha^2 \int_{-\pi}^{\pi} \frac{|k(z) - k(\bar{z})|^2}{|1 - z|^4} dt$$

or, equivalently,

$$\int_{-\pi}^{\pi} \frac{|f(z)|^2}{|1 - z|^4} dt - \alpha^2 \int_{-\pi}^{\pi} \frac{|k(z)|^2}{|1 - z|^4} dt \geqq \operatorname{Re} \int_{-\pi}^{\pi} \frac{g(z)}{|1 - z|^4} dt$$

where

$$g(z) = f(z)\overline{f(\bar{z})} - \alpha^2 k(z)^2.$$

Hence, (29) will hold provided we can prove that

$$\text{Re} \int_{-\pi}^{\pi} \frac{g(re^{it})}{|1 - re^{it}|^4} dt \geqq 0, \qquad 0 < r < 1. \tag{30}$$

Notice that, since f has real coefficients, $\overline{f(\bar{z})} = f(z)$ and therefore

$$g(z) = f(z)^2 - \alpha^2 k(z)^2. \tag{31}$$

Define

$$F(z) = \frac{zg(z)}{(1-z)^2} = g(z)k(z). \tag{32}$$

We have

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{g(re^{it})}{|1 - re^{it}|^4} dt &= \int_{-\pi}^{\pi} \frac{g(re^{it})r^2 e^{2it}}{(1 - re^{it})^2 (re^{it} - r^2)^2} dt \\
&= \frac{1}{i} \int_{|z|=r} \frac{zg(z)}{(1-z)^2 (z - r^2)^2} dz \\
&= \frac{1}{i} \int_{|z|=r} \frac{F(z)}{(z - r^2)^2} dz \\
&= 2\pi \,\text{Res}\left(\frac{F(z)}{(z - r^2)^2}, r^2\right) \\
&= 2\pi F'(r^2).
\end{aligned}$$

Notice that (31) and (32) show that

$$F' = 2(ff' - \alpha^2 kk')k + (f^2 - \alpha^2 k^2)k'. \tag{33}$$

Now, since f has real coefficients, $|f'(r^2)| = f'(r^2)$ and $|f(r^2)| = f(r^2)$ which, together with (2), (28), (33) and the assumption $\theta_0 = 0$, implies $F'(r^2) \geqq 0$. This proves (30) finishing the proof of Theorem 3.

Next we shall prove that the inequality (5) remains true for $2 < p < 3$.

THEOREM 4. *Let $0 < \alpha \leqq 1$ and $f \in S(\alpha)$, then, for $2 < p < 3$,*

$$M_p(r, f) \geqq M_p(r, \alpha k), \qquad 0 < r < 1.$$

Theorem 4 will follow from the following

PROPOSITION 2. *Let $0 < \alpha \leqq 1$ and $f \in S(\alpha)$. Suppose that $\theta_0 = 0$ is the direction of maximal growth of f. Then*

(34) $$\alpha^2 \int_{-\pi}^{\pi} |k(re^{it})|^{2+q}dt \leqq \int_{-\pi}^{\pi} |f(re^{it})|^2|k(re^{it})|^q dt, \qquad 0<r<1,$$

for every q *with* $0<q<1$.

PROOF OF PROPOSITION 2. Arguing as in the proof of Theorem 3 we see that (34) will hold provided we can show that, for $0<q<1$,

(35) $$\int_{-\pi}^{\pi} \frac{g(re^{it})}{|1-re^{it}|^{2q}}dt \geqq 0, \qquad 0<r<1,$$

where

(36) $$g(z)=f(z)\overline{f(\bar{z})}-\alpha^2 k(z)^2.$$

Now, it is easy to check that

(37) $$\int_{-\pi}^{\pi} \frac{g(re^{it})}{|1-re^{it}|^{2q}}dt = \frac{1}{i}\int_{|z|=r} \frac{g(z)z^{q-1}}{(1-z)^q(z-r^2)^q}dz.$$

In order to compute the integral on the right-hand side of (37), notice that, for $\delta>0$ sufficiently small,

$$\int_{C_\delta} \frac{g(z)z^{q-1}}{(1-z)^q(z-r^2)^q}dz = 0$$

where C_δ is the contour of Fig. 1.

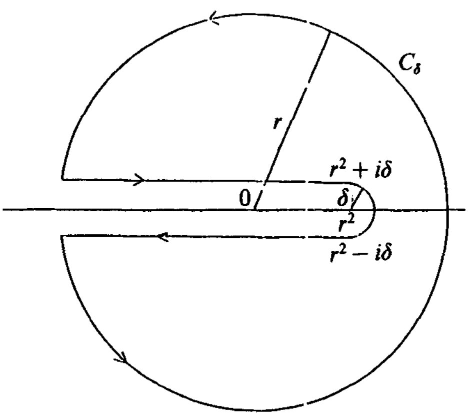

Fig. 1.

It is a simple exercise to prove that, for $0 < q < 1$,

$$0 = \lim_{\delta\to 0} \frac{1}{i} \int_{C_\delta} \frac{g(z)z^{q-1}}{(1-z)^q(z-r^2)^q}\, dz$$

$$= \frac{1}{i} \int_{|z|=r} \frac{g(z)z^{q-1}}{(1-z)^q(z-r^2)^q}\, dz - 2\sin(\pi q) \int_0^{r^2} \frac{g(s)s^{q-1}}{(1-s)^q(r^2-s)^q}\, ds.$$

Therefore

$$\frac{1}{i} \int_{|z|=r} \frac{g(z)z^{q-1}}{(1-z)^q(z-r^2)^q}\, dz = 2\sin(\pi q) \int_0^{r^2} \frac{g(s)s^{q-1}}{(1-s)^q(r^2-s)^q}\, ds. \tag{38}$$

Now, (36) and (2) show that if $s > 0$ then $g(s) \geqq 0$ which, since $0 < q < 1$, implies that the right-hand side of (38) is nonnegative. This proves (35) and hence Proposition 2 is proved.

Proof of Theorem 4. We may assume without loss of generality that $\theta_0 = 0$ is the direction of maximal growth of f. Then, if $2 < p < 3$, we have by Proposition 2 and Hölder's inequality

$$\alpha^2 \int_{-\pi}^{\pi} |k(re^{it})|^p dt \leqq \int_{-\pi}^{\pi} |f(re^{it})|^2 |k(re^{it})|^{p-2} dt$$

$$\leqq \left(\int_{-\pi}^{\pi} |f(re^{it})|^p dt \right)^{2/p} \left(\int_{-\pi}^{\pi} |k(re^{it})|^p dt \right)^{(p-2)/p}$$

and hence

$$\alpha^2 \left(\int_{-\pi}^{\pi} |k(re^{it})|^p dt \right)^{2/p} \leqq \left(\int_{-\pi}^{\pi} |f(re^{it})|^p dt \right)^{2/p}$$

which proves the Theorem.

4. Final remarks

(i) First we note that Kamotskii's result for $p = 1$ can be obtained in an elementary way as a consequence of the following inequality proved by Holland and Twomey in [9, p. 1018]:

If g is analytic in D, then, for $0 < p < \infty$,

$$(1 - r^2) M_\infty(r^2, g) \leqq M_p(r, g)^p, \qquad 0 < r < 1. \tag{39}$$

Now, if $f \in S(\alpha)$ set $g(z) = f(z)/z$. Then $M_\infty(r^2, g) \geqq \alpha k(r^2)/r^2$ and (39) with $p = 1$ shows that

$$M_1(r,g) \geqq (1-r^2)\frac{\alpha k(r^2)}{r^2} = \frac{\alpha}{1-r^2} = \frac{\alpha}{2\pi}\int_{-\pi}^{\pi}\left|\frac{k(re^{it})}{r}\right| dt$$

which implies $M_1(r,f) \geqq M_1(r,\alpha k)$.

This argument can be used for other values of p and also to estimate the integral means of f'. In this way we obtain:

If $f \in S(\alpha)$ then, for $0<r<1$,

(40) $$rM_p(r,f) \geqq M_p(r^2,\alpha k), \qquad p>1;$$

(41) $$M_p(r,f') \geqq M_p(r^2,\alpha k'), \qquad p \geqq 1.$$

(ii) The only place where we used that f had real Taylor coefficients in the proof of Theorem 3 was to prove, with the notation used there, that $F'(r^2) \geqq 0$ which followed from $g'(s) \geqq 0$, $0<s<1$. This, and hence also the conclusion of Theorem 3, would be true for general $f \in S(\alpha)$, not necessarily with real coefficients, if the inequality (28) could be replaced by

$$\frac{d}{dr}\,|f(re^{i\theta_0})| \geqq \alpha k'(r), \qquad 0<r<1,$$

but we do not know whether or not this is true.

(iii) Even though Theorem 2 shows that the conclusion of Theorem 1 does not hold for general $f \in S(\alpha)$, the question of whether or not the inequality $M_p(r,f) \geqq M_p(r,\alpha k)$ is true for all $p>0$ remains open. We can prove that it is true for small values of p using the elementary fact that if h is analytic in D and $|h(z)|>1$ for all $z \in D$ then, given $p>0$, the function $|h|^p \log|h|$ is subharmonic in D. In a precise way, we have:

Given $0<\alpha<1$ there exists $p(\alpha)>0$ such that if $f \in S(\alpha)$ and $0<p<p(\alpha)$ then

$$M_p(r,f) \geqq M_p(r,\alpha k), \qquad 0<r<1.$$

Indeed, let $0<\alpha<1$ and $f \in S(\alpha)$. The distortion theorem [15, p. 21] shows that

$$\left|\frac{4f(z)}{\alpha z}\right| > \frac{1}{\alpha} > 1, \quad \left|\frac{4k(z)}{z}\right| > 1, \qquad z \in D$$

and therefore, for $p > 0$, the functions

$$u(z) = \left|\frac{4f(z)}{\alpha z}\right|^p \log\left|\frac{4f(z)}{\alpha z}\right|, \qquad v(z) = \left|\frac{4k(z)}{z}\right|^p \log\left|\frac{4k(z)}{z}\right|$$

are subharmonic in D. For $0 < r < 1$, define

$$h_r(p) = \int_{-\pi}^{\pi} \left|\frac{4f(re^{it})}{\alpha r}\right|^p dt - \int_{-\pi}^{\pi} \left|\frac{4k(re^{it})}{r}\right|^p dt, \qquad p > 0.$$

Then

$$\frac{d}{dp} h_r(p) = \int_{-\pi}^{\pi} u(re^{it})dt - \int_{-\pi}^{\pi} v(re^{it})dt.$$

Hence, since $S \subset H^p$ $(p < \frac{1}{2})$, the subharmonicity of u and v shows that, for $0 < p < \frac{1}{2}$,

$$\frac{d}{dp} h_r(p) \geqq 2\pi \left(\frac{4}{\alpha}\right)^p \log\frac{4}{\alpha} - \int_{-\pi}^{\pi} |4k(e^{it})|^p \log|4k(e^{it})|\, dt. \tag{42}$$

Now, as $p \to 0$, the right-hand side of (42) tends to $2\pi \log(1/\alpha) > 0$. Hence we deduce that there exists $p(\alpha) > 0$ such that, for $0 < p < p(\alpha)$,

$$\frac{d}{dp} h_r(p) > 0, \qquad 0 < r < 1,$$

which, together with the fact $h_r(0) = 0$, implies that, for $0 < r < 1$,

$$h_r(p) > 0, \qquad 0 < p < p(\alpha),$$

which proves our assertion.

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